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Multiple positive solutions for a class of m-point boundary value problems[☆]

Xuemei Zhang[a,c], Meiqiang Feng[b,c,∗], Weigao Ge[c]

[a] *Department of Mathematics and Physics, North China Electric Power University, Beijing 102206, PR China*
[b] *Department of Fundamental Sciences, Beijing Information Technology Institute, Beijing, 100101, PR China*
[c] *Department of Mathematics, Beijing Institute of Technology, Beijing, 100081, PR China*

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Abstract

Using a specially constructed cone and the fixed point index theory, this work shows the existence of multiple positive solutions for a class of m-point Sturm–Liouville boundary value problems for second-order differential equations $\lambda x'' + g(t)f(t,x) = 0$, where $\lambda > 0$ and $g(t)$ is L^p-integrable.

Keywords: Multi-point boundary value problem; Positive solution; Completely continuous; Fixed point index theory; Existence

1. Introduction

In this work, we study the existence of multiplicity of positive solutions for the following second-order m-point boundary value problem (BVP):

$$\begin{cases} \lambda x'' + g(t)f(t,x) = 0, \quad 0 < t < 1, \\ ax(0) - bx'(0) = \sum_{i=1}^{m-2} a_i x(\xi_i), \\ cx(1) + dx'(1) = \sum_{i=1}^{m-2} b_i x(\xi_i), \end{cases} \tag{1.1$_\lambda$}$$

where $\lambda > 0$, $a \geq 0, b \geq 0, c \geq 0, d \geq 0, \rho := ac + bc + ad > 0$, $\xi_i \in (0,1), a_i, b_i \in (0,+\infty)(i = 1,2,\ldots,m-2)$, $g \in L^p[0,1]$ for some $1 \leq p \leq +\infty$, $f \in C([0,1] \times [0,+\infty), [0,+\infty))$.

Multi-point boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problem; many problems in the theory of

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∗ Corresponding author at: Department of Fundamental Sciences, Beijing Information Technology Institute, Beijing, 100101, PR China.
E-mail address: meiqiangfeng@sina.com (M. Feng).

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elastic stability can be handled as multi-point boundary value problems too. Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations and difference equations have received a great deal of attention. To identify a few cases, we refer the reader to [1,5–15] and references therein.

Motivated by the results mentioned above, in this work we study the multiplicity of positive solutions for BVP $(1.1)_\lambda$ using a new technique (different from that of the proof of Theorem 5.5.1 of [5]) to overcome difficulties arising from the m-point appearances and $g(t)$ being L^p-integrable. The arguments are based upon the fixed point index theory.

Fixed point index theorems have been applied to various boundary value problems to show the existence of multiple positive solutions. An overview of such results can be found in Guo and Lakshmikantham [2], in Guo, Lakshmikantham and Liu [3], in Guo [4], in Deimling [16] and in Krasnoselskii [17].

Lemma 1.1 (*[2–4,16,17]*). *Let E be a real Banach space and K be a cone in E. For $r > 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{x \in K : \|x\| = r\}$.*

(i) *If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

(ii) *If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1.$$

To obtain some of the norm inequalities in Theorem 3.1 and Corollary 3.1, we employ Hölder's inequality.

Lemma 1.2 (*Hölder*). *Let $f \in L^p[a, b]$ with $p > 1$, $g \in L^q[a, b]$ with $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1[a, b]$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Let $f \in L^1[a, b]$, $g \in L^\infty[a, b]$. Then $fg \in L^1[a, b]$ and

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty.$$

The work is organized in the following fashion. In Section 2, we provide some necessary background. In particular, we state some properties of the Green's function associated with BVP $(1.1)_\lambda$. In Section 3, the main results will be stated and proved.

2. Preliminaries

Let $J = [0, 1]$. The basic space used in this work is $E = C[0, 1]$. It is well known that E is a real Banach space with the norm $\|\cdot\|$ defined by $\|x\| = \max_{0 \leq t \leq 1} |x(t)|$. Let K be a cone of E, $K_r = \{x \in K : \|x\| \leq r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$, $K_\gamma^R = \{x \in K : \|x\| < R, \min_{t \in J_\theta} x(t) > \gamma\}$, $\bar{K}_\gamma^R = \{x \in K : \|x\| \leq R, \min_{t \in J_\theta} x(t) \geq \gamma\}$, $\partial K_\gamma^R = \{x \in K : \|x\| = R, \min_{t \in J_\theta} x(t) = \gamma\}$, where $\theta \in (0, \frac{1}{2})$, $J_\theta = [\theta, 1 - \theta]$, $r > 0$, $R > 0$, $\gamma > 0$.

The following assumptions will stand throughout this work:

(H_1) $g \in L^p[0, 1]$ for some $1 \leq p \leq +\infty$ and there exists $m > 0$ such that $g(t) \geq m$ a.e. on J;
(H_2) $f \in C(J \times [0, +\infty), [0, +\infty))$ and $f(t, 0) = 0$ uniformly with respect to t on J;
(H_3) $\Delta < 0$, $\rho - \sum_{i=1}^{m-2} a_i \phi(\xi_i) > 0$, $\rho - \sum_{i=1}^{m-2} b_i \psi(\xi_i) > 0$,

where

$$\Delta = \begin{vmatrix} -\sum_{i=1}^{m-2} a_i \psi(\xi_i) & \rho - \sum_{i=1}^{m-2} a_i \phi(\xi_i) \\ \rho - \sum_{i=1}^{m-2} b_i \psi(\xi_i) & -\sum_{i=1}^{m-2} b_i \phi(\xi_i) \end{vmatrix},$$

$$\psi(t) = b + at, \quad \phi(t) = c + d - ct, \quad t \in J \tag{2.1}$$

are linearly independent solutions of the equation $x'' = 0$.

We remark that (2.1) shows that ψ is nondecreasing on J and ϕ is nonincreasing on J.
For the sake of applying Lemma 1.1, we construct a cone K in E using

$$K = \{x \in E : x(t) \geq 0, t \in J\}. \tag{2.2}$$

It is easy to see that K is a closed convex cone of E.
Define $T_\lambda : K \to K$ by

$$(T_\lambda x)(t) = \frac{1}{\lambda}\left[\int_0^1 G(t,s)g(s)f(s,x(s))\mathrm{d}s + A(g(\cdot)f(\cdot,x(\cdot)))\psi(t) + B(g(\cdot)f(\cdot,x(\cdot)))\phi(t)\right], \tag{2.3}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases}\psi(s)\phi(t), & \text{if } 0 \leq s \leq t \leq 1,\\ \psi(t)\phi(s), & \text{if } 0 \leq t \leq s \leq 1,\end{cases} \tag{2.4}$$

$$A(gy) := \frac{1}{\Delta}\begin{vmatrix}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,t)g(t)y(t)\mathrm{d}t & \rho - \sum_{i=1}^{m-2} a_i\phi(\xi_i)\\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,t)g(t)y(t)\mathrm{d}t & -\sum_{i=1}^{m-2} b_i\phi(\xi_i)\end{vmatrix}, \tag{2.5}$$

$$B(gy) := \frac{1}{\Delta}\begin{vmatrix}-\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,t)g(t)y(t)\mathrm{d}t\\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,t)g(t)y(t)\mathrm{d}t\end{vmatrix}. \tag{2.6}$$

It is not difficult to show that $G(t,s)$, $A(y)$ and $B(y)$ have the following properties.

Proposition 2.1. *For $t, s \in (0,1)$, we have*

$$G(t,s) > 0. \tag{2.7}$$

Proposition 2.2. *For $t, s \in J$, we have*

$$0 \leq G(t,s) \leq G(s,s). \tag{2.8}$$

Proposition 2.3. *For all $t \in J_\theta$, $s \in (0,1)$, we have*

$$G(t,s) \geq \sigma G(s,s), \tag{2.9}$$

where

$$\sigma = \sigma_\theta = \min\left\{\frac{\psi(1-\theta)}{\psi(0)}, \frac{\phi(\theta)}{\phi(1)}\right\}. \tag{2.10}$$

In fact, for $t \in [\theta, 1-\theta]$, we have

$$\frac{G(t,s)}{G(s,s)} \geq \min\left\{\frac{\psi(1-\theta)}{\psi(s)}, \frac{\phi(\theta)}{\phi(s)}\right\} \geq \min\left\{\frac{\psi(1-\theta)}{\psi(0)}, \frac{\phi(\theta)}{\phi(1)}\right\} =: \sigma.$$

It is easy to see that $0 < \sigma < 1$.

Proposition 2.4. *From (2.5), we have*

$$|A(gy)| \leq \frac{1}{\Delta}\begin{vmatrix}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,t)g(t)\mathrm{d}t & \rho - \sum_{i=1}^{m-2} a_i\phi(\xi_i)\\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,t)g(t)\mathrm{d}t & -\sum_{i=1}^{m-2} b_i\phi(\xi_i)\end{vmatrix} \|y\| := \tilde{A}\|y\|, \tag{2.11}$$

Proposition 2.5. *From (2.6), we have*

$$|B(gy)| \leq \frac{1}{\Delta} \begin{vmatrix} -\sum_{i=1}^{m-2} a_i \psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, t) g(t) \mathrm{d}t \\ \rho - \sum_{i=1}^{m-2} b_i \psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, t) g(t) \mathrm{d}t \end{vmatrix} \|y\| := \tilde{B} \|y\|. \tag{2.12}$$

We remark that Proposition 2.3 implies that there exists $\tau > 0$ such that for $t, s \in J_\theta$

$$G(t, s) \geq \tau. \tag{2.13}$$

It is easy to obtain the following Lemma 2.1 by (2.3).

Lemma 2.1 (*[5]*). *Let* $(\mathrm{H}_1)-(\mathrm{H}_3)$ *hold. Then BVP* $(1.1)_\lambda$ *has a solution* x *if and only if* x *is a fixed point of* T_λ.

The proof is similar to that of Lemma 5.5.1 of [5].

Lemma 2.2. *Let* $(\mathrm{H}_1)-(\mathrm{H}_3)$ *hold. Then* $T_\lambda K \subset K$ *and* $T_\lambda : K \to K$ *is completely continuous.*

Proof. For any $x \in K$, by (2.3), we obtain $T_\lambda x \geq 0$. Next by standard methods and the Ascoli–Arzela theorem one can prove that $T_\lambda : K \to K$ is completely continuous. So this is omitted. □

3. Main results

In this section, we apply Lemmas 1.1 and 1.2 to establish the multiplicity of positive solutions for BVP $(1.1)_\lambda$. We consider the following three cases for $g \in L^p[0, 1]$: $p > 1$, $p = 1$, and $p = \infty$. Case $p > 1$ is treated in the following theorem.

Theorem 3.1. *Assume that* $(\mathrm{H}_1)-(\mathrm{H}_3)$ *are satisfied. In addition, let* f *satisfy the following conditions:*

(H_4) $\lim_{x\to 0} \frac{f(t,x)}{x} = 0$ *uniformly with respect to* $t \in J$;

(H_5) $\lim_{x\to +\infty} \frac{f(t,x)}{x} = 0$ *uniformly with respect to* $t \in J$;

(H_6) *There exists* $\gamma > 0$, *for* $x \geq \gamma$, $t \in J$, *such that* $f(t, x) \geq \eta$, *where* $\eta > 0$; *then there exists* $\delta > 0$ *such that, for* $0 < \lambda < \delta$, *BVP* $(1.1)_\lambda$ *has at least two positive solutions* $x_\lambda^{(1)}(t)$, $x_\lambda^{(2)}(t)$ *and* $\max_{t\in J} x_\lambda^{(1)}(t) > \gamma$.

Proof. Let $\delta = \tau m \eta (1 - 2\theta) \gamma^{-1}$; then for $0 < \lambda < \delta$, (2.3) and Lemma 2.2 imply that $T_\lambda : K \to K$ is completely continuous.

Considering (H_4), there exists $0 < r < \gamma$ such that $f(t, x) \leq \frac{\lambda}{2\Lambda} x$, for $0 \leq x \leq r$, $t \in J$, where $\Lambda = \|G\|_q \|g\|_p + \tilde{A}\|\psi\| + \tilde{B}\|\phi\|$.

So, for $x \in \partial K_r$, we have from (2.9)

$$\begin{aligned} (T_\lambda x)(t) &= \frac{1}{\lambda} \left[\int_0^1 G(t, s) g(s) f(s, x(s)) \mathrm{d}s + A(g(\cdot) f(\cdot, x(\cdot)))\psi(t) + B(g(\cdot) f(\cdot, x(\cdot)))\phi(t) \right] \\ &\leq \frac{1}{\lambda} \left[\int_0^1 G(t, s) g(s) \mathrm{d}s + \tilde{A}\|\psi\| + \tilde{B}\|\phi\| \right] \|f(t, x)\| \\ &\leq \frac{1}{\lambda} \left[\int_0^1 G(t, s) g(s) \mathrm{d}s + \tilde{A}\|\psi\| + \tilde{B}\|\phi\| \right] \frac{\lambda}{2\Lambda} \|x\| \\ &\leq \frac{1}{\lambda} \left[\int_0^1 G(s, s) g(s) \mathrm{d}s + \tilde{A}\|\psi\| + \tilde{B}\|\phi\| \right] \frac{\lambda}{2\Lambda} \|x\| \\ &\leq \frac{1}{\lambda} \left[\|G\|_q \|g\|_p + \tilde{A}\|\psi\| + \tilde{B}\|\phi\| \right] \frac{\lambda}{2\Lambda} \|x\| \\ &= \frac{\|x\|}{2} < \|x\| = r. \end{aligned} \tag{3.1}$$

Consequently, for $x \in \partial K_r$, we have $\|T_\lambda x\| < \|x\|$, i.e., by Lemma 1.1,

$$i(T_\lambda, K_r, K) = 1. \tag{3.2}$$

Now turning to (H_5), there exists $m > 0$, for $t \in J, x > m$, such that $f(t,x) \le \frac{\lambda}{2\Lambda}x$. Letting $\rho = \max_{t\in J, 0\le x\le m} f(t,x)$, then

$$0 \le f(t,x) \le \frac{\lambda}{2\Lambda}x + \rho. \tag{3.3}$$

Choose

$$R > \max\left\{\mu, 2\frac{\Lambda\rho}{\lambda}\right\}. \tag{3.4}$$

So for $x \in \partial K_R$, from (3.3) and (3.4) we have

$$\begin{aligned}
(T_\lambda x)(t) &= \frac{1}{\lambda}\left[\int_0^1 G(t,s)g(s)f(s,x(s))\mathrm{d}s + A(g(\cdot)f(\cdot,x(\cdot)))\psi(t) + B(g(\cdot)f(\cdot,x(\cdot)))\phi(t)\right] \\
&\le \frac{1}{\lambda}\left[\int_0^1 G(t,s)g(s)\mathrm{d}s + \tilde{A}\|\psi\| + \tilde{B}\|\phi\|\right]\|f(t,x)\| \\
&\le \frac{1}{\lambda}\left[\int_0^1 G(t,s)g(s)\mathrm{d}s + \tilde{A}\|\psi\| + \tilde{B}\|\phi\|\right]\left(\frac{\lambda}{2\Lambda} + \rho\right)\|x\| \\
&\le \frac{1}{\lambda}\left[\int_0^1 G(s,s)g(s)\mathrm{d}s + \tilde{A}\|\psi\| + \tilde{B}\|\phi\|\right]\left(\frac{\lambda}{2\Lambda} + \rho\right)\|x\| \\
&\le \frac{1}{\lambda}\left[\|G\|_q\|g\|_p + \tilde{A}\|\psi\| + \tilde{B}\|\phi\|\right]\left(\frac{\lambda}{2\Lambda} + \rho\right)\|x\| \\
&< \frac{\|x\|}{2} + \frac{R}{2} = \|x\|,
\end{aligned}$$

i.e., by Lemma 1.1,

$$i(T_\lambda, K_R, K) = 1. \tag{3.5}$$

On the other hand, for $x \in \bar{K}_\gamma^R = \{x \in K : \|x\| \le R, \min_{t\in J_\theta} x(t) \ge \gamma\}$, $t \in J$, (2.3), (2.11) and (2.12) yield that

$$\|T_\lambda x\| \le \frac{1}{\lambda}\left[\|G\|_q\|g\|_p + \tilde{A}\|\psi\| + \tilde{B}\|\phi\|\right]\left(\frac{\lambda}{2\Lambda} + \rho\right)\|x\| < R. \tag{3.6}$$

Furthermore, for $x \in \bar{K}_\gamma^R$, from (2.3) and (2.9) and (H_6), we obtain

$$\begin{aligned}
\min_{t\in J_\theta}(T_\lambda x)(t) &= \min_{t\in J_\theta}\frac{1}{\lambda}\left[\int_0^1 G(t,s)g(s)f(s,x(s))\mathrm{d}s + A(g(\cdot)f(\cdot,x(\cdot)))\psi(t) + B(g(\cdot)f(\cdot,x(\cdot)))\phi(t)\right] \\
&\ge \min_{t\in J_\theta}\frac{1}{\lambda}\int_0^1 G(t,s)g(s)f(s,x(s))\mathrm{d}s \\
&\ge \min_{t\in J_\theta}\frac{1}{\lambda}\int_\theta^{1-\theta} G(t,s)g(s)f(s,x(s))\mathrm{d}s \\
&\ge \frac{1}{\lambda}\tau\int_\theta^{1-\theta} g(s)f(s,x(s))\mathrm{d}s \\
&\ge \frac{1}{\lambda}\tau m\eta(1-2\theta)
\end{aligned}$$

$$\begin{aligned} &> \frac{1}{\delta}\tau m\eta(1-2\theta) \\ &= \gamma. \end{aligned} \tag{3.7}$$

Let $x_0 \equiv \frac{\gamma+R}{2}$ and $H(t,x) = (1-t)T_\lambda x + tx_0$; then $H : [0,1] \times \bar{K}_\gamma^R \to K$ is completely continuous, and from the analysis above, we obtain for $(t,x) \in [0,1] \times \bar{K}_\gamma^R$

$$H(t,x) \in K_\gamma^R. \tag{3.8}$$

Therefore, for $t \in J$, $x \in \partial K_\gamma^R$, we have $H(t,x) \neq x$. Hence, by the normality property and the homotopy invariance property of the fixed point index, we obtain

$$i(T_\lambda, K_\gamma^R, K) = i(x_0, K_\gamma^R, K) = 1. \tag{3.9}$$

Consequently, by the solution property of the fixed point index, T_λ has a fixed point $x_\lambda^{(1)}$ and $x_\lambda^{(1)} \in K_\gamma^R$. By Lemma 2.1, it follows that $x_\lambda^{(1)}$ is a solution to BVP $(1.1)_\lambda$, and

$$\max_{t\in J} x_\lambda^{(1)} \geq \min_{t\in J_\theta} x_\lambda^{(1)} > \gamma.$$

On the other hand, from (3.2), (3.3) and (3.7) together with the additivity of the fixed point index, we get

$$\begin{aligned} i(T_\lambda, K_R \setminus (\bar{K}_r \cup \bar{K}_\gamma^R), K) &= i(T_\lambda, K_R, K) - i(T_\lambda, K_\gamma^R, K) - i(T_\lambda, K_r, K) \\ &= 1 - 1 - 1 = -1. \end{aligned} \tag{3.10}$$

Hence, by the solution property of the fixed point index, T_λ has a fixed point $x_\lambda^{(2)}$ and $x_\lambda^{(2)} \in K_R \setminus (\bar{K}_r \cup \bar{K}_\gamma^R)$. By Lemma 2.1, it follows that $x_\lambda^{(2)}$ is also a solution to BVP $(1.1)_\lambda$, and $x_\lambda^{(1)} \neq x_\lambda^{(2)}$. The proof is complete. □

The proofs of the remaining theorems in this section are similar to the proof of Theorem 3.1. We will present only sketches of their proofs.

The following theorem deals with the case $p = \infty$.

Corollary 3.1. *Assume that* (H_1)–(H_6) *are satisfied. Then there exists* $\delta > 0$ *such that, for* $0 < \lambda < \delta$, *BVP* $(1.1)_\lambda$ *has at least two positive solutions* $x_\lambda^{(1)}(t)$, $x_\lambda^{(2)}(t)$ *and* $\max_{t\in J} x_\lambda^{(1)}(t) > \gamma$.

Proof. Let $\|G\|_1\|g\|_\infty$ replace $\|G\|_p\|g\|_q$ and repeat the argument above. □

Finally we consider the case of $p = 1$.

Corollary 3.2. *Assume that* (H_1)–(H_6) *are satisfied. Then there exists* $\delta > 0$ *such that, for* $0 < \lambda < \delta$, *BVP* $(1.1)_\lambda$ *has at least two positive solutions* $x_\lambda^{(1)}(t)$, $x_\lambda^{(2)}(t)$ *and* $\max_{t\in J} x_\lambda^{(1)}(t) > \gamma$.

Proof. For $x \in \partial K_r$, we have from (2.9)

$$\begin{aligned} (T_\lambda x)(t) &= \frac{1}{\lambda}\left[\int_0^1 G(t,s)g(s)f(s,x(s))\mathrm{d}s + A(g(\cdot)f(\cdot,x(\cdot)))\psi(t) + B(g(\cdot)f(\cdot,x(\cdot)))\phi(t)\right] \\ &\leq \frac{1}{\lambda}\left[\int_0^1 G(t,s)g(s)\mathrm{d}s + \tilde{A}\|\psi\| + \tilde{B}\|\phi\|\right]\|f(t,x)\| \\ &\leq \frac{1}{\lambda}\left[\int_0^1 G(t,s)g(s)\mathrm{d}s + \tilde{A}\|\psi\| + \tilde{B}\|\phi\|\right]\frac{\lambda}{2\Lambda}\|x\| \\ &\leq \frac{1}{\lambda}\left[\int_0^1 G(s,s)g(s)\mathrm{d}s + \tilde{A}\|\psi\| + \tilde{B}\|\phi\|\right]\frac{\lambda}{2\Lambda}\|x\| \\ &\leq \frac{1}{\lambda}\left[\frac{1}{\rho}\psi(1)\phi(0)\|g\|_p + \tilde{A}\|\psi\| + \tilde{B}\|\phi\|\right]\frac{\lambda}{2\Lambda'}\|x\| \end{aligned}$$

$$= \frac{\|x\|}{2} < \|x\| = r,$$

where $\Lambda' = \frac{1}{\rho}\psi(1)\phi(0)\|g\|_p + \tilde{A}\|\psi\| + \tilde{B}\|\phi\|$.

Consequently, for $x \in \partial K_r$, we have $\|T_\lambda x\| < \|x\|$, i.e., by Lemma 1.1,

$$i(T_\lambda, K_r, K) = 1. \tag{3.2$'$}$$

Similarly, if $x \in \partial K_R$ we can obtain

$$i(T_\lambda, K_R, K) = 1. \tag{3.5$'$}$$

If $x \in \bar{K}_\gamma^R$, we also obtain

$$\|T_\lambda x\| \leq \frac{1}{\lambda}\left[\frac{1}{\rho}\psi(1)\phi(0)\|g\|_p + \tilde{A}\|\psi\| + \tilde{B}\|\phi\|\right]\left(\frac{\lambda}{2\Lambda'} + \rho\right)\|x\| < R. \tag{3.6$'$}$$

Like in the proof of Theorem 3.1, (3.2), (3.3), (3.9) and (3.10) also follow from the results of Corollary 3.2. □

Remark 3.1. In Theorem 3.1, Corollaries 3.1 and 3.2, we generalize the Theorem 5.5.1 of [5] in three main directions:

(1) The parameter $\lambda > 0$ is considered.
(2) $g(t)$ is L^p-integrable.
(3) We need $\lim_{x\to 0}\frac{f(t,x)}{x} = 0$ and $\lim_{x\to+\infty}\frac{f(t,x)}{x} = 0$ uniformly with respect to $t \in J$, but Theorem 5.5.1 of [5] needs f to be either superlinear or sublinear which is the key condition in Theorem 5.5.1 of [5].

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